

The Sagnac Phase Shift suggested by the Aharonov-Bohm effect for relativistic matter beams

Guido Rizzi^{§,¶} and Matteo Luca Ruggiero^{§,¶,*}

[§] Dipartimento di Fisica, Politecnico di Torino,

[¶] INFN, Sezione di Torino

E-mail guido.rizzi@polito.it, matteo.ruggiero@polito.it

February 7, 2008

Abstract

The phase shift due to the Sagnac Effect, for relativistic matter beams counter-propagating in a rotating interferometer, is deduced on the bases of a formal analogy with the Aharonov-Bohm effect. A procedure outlined by Sakurai, in which non relativistic quantum mechanics and newtonian physics appear together with some intrinsically relativistic elements, is generalized to a fully relativistic context, using the Cattaneo's splitting technique. This approach leads to an exact derivation, in a self-consistently relativistic way, of the Sagnac effect. Sakurai's result is recovered in the first order approximation.

Keywords: Sagnac Effect, Aharonov-Bohm Effect, Special Relativity, non-time-orthogonal frames.

1 Introduction

1.1 The early years

The story of the interferometrical detection of the effects of rotation dates back to the end of the XIX century when, still in the context of the ether theory, Sir Oliver Lodge[1] proposed to use a large interferometer to detect the rotation of the Earth. Subsequently[2] he proposed to use an interferometer rotating on a turntable in order to reveal rotation effects with respect

*Dipartimento di Fisica del Politecnico di Torino, Corso Duca degli Abruzzi 24, I-10129 Torino - Italy, tel. +390115647380, fax +390115647399

to the laboratory frame. A detailed description of these early works can be found in the paper by Anderson *et al.*[3], where the study of rotating interferometers is analyzed in a historical perspective. In 1913 Sagnac[4] verified his early predictions[5], using a rapidly rotating light-optical interferometer. In fact, on the ground of classical physics, he predicted the following fringe shift (with respect to the interference pattern when the device is at rest), for monochromatic light waves in vacuum, counter-propagating along a closed path in a rotating interferometer:

$$\Delta z = \frac{4\mathbf{\Omega} \cdot \mathbf{S}}{\lambda c} \quad (1)$$

where $\mathbf{\Omega}$ is the (constant) angular velocity vector of the turntable, \mathbf{S} is the vector associated to the area enclosed by the light path, and λ is the wavelength of light in vacuum. The time difference associated to the fringe shift (1) turns out to be

$$\Delta t = \frac{\lambda}{c} \Delta z = \frac{4\mathbf{\Omega} \cdot \mathbf{S}}{c^2} \quad (2)$$

Even if his interpretation of these results was entirely in the framework of the classical (non Lorentz!) ether theory, Sagnac was the first scientist who reported an experimental observation of the effect of rotation on spacetime, which, after him, was named "Sagnac effect". It is interesting to notice that the Sagnac effect was interpreted as a disproof of the Special Theory of Relativity (SRT) not only during the early years of relativity (in particular by Sagnac himself), but, also, more recently, in the 90's by Selleri[6],[7], Croca-Selleri[8], Goy-Selleri[9], Vigier[10], Anastasovski *et al.*[11]. However, this claim is incorrect: the Sagnac effect can be explained completely in the framework of SRT, which allows a deeper insight into its very foundations. In fact, it can be interpreted as an observable consequence of the synchronization gap predicted by SRT for non-time-orthogonal physical frames (see Weber[12], Dieks[13], Anandan[14], Rizzi-Tartaglia[15], Bergia-Guidone [16], Rodrigues-Sharif[17]). In particular, SRT predicts the following proper time difference (as measured by a clock at rest in the starting/ending point on the turntable) between light beams counter-propagating in a ring interferometer

$$\Delta\tau = \frac{4\pi R^2\Omega}{c^2 \left(1 - \frac{\Omega^2 R^2}{c^2}\right)^{1/2}} \quad (3)$$

where R is the radius of the ring. Evidently, relation (3) reduces to (2) in the first order approximation (with respect to the small parameter $\frac{\Omega R}{c}$).

Few years before Sagnac, Franz Harres[18], graduate student in Jena, observed, for the first time but unknowingly, the Sagnac effect during his experiments on the Fresnel-Fizeau drag of light. However, only in 1914, Harzer[19] recognized that the unexpected and inexplicable bias found by Harres was nothing else than the manifestation of the Sagnac effect. Moreover, Harres's observations also demonstrated that the Sagnac fringe shift is unaffected by refraction: in other words, it is always given by eq. (1), provided that λ is interpreted as the light wavelength in a comoving refractive medium. So, the Sagnac phase shift depends on the light wavelength, and not on the velocity of light in the (comoving) medium.

If Harres anticipated the Sagnac effect on the experimental ground, Michelson[20] anticipated the effect on the theoretical side. Subsequently, in 1925, Michelson himself and Gale[21] succeeded in measuring a phase shift, analogous to the Sagnac's one, caused by the rotation of the Earth, using a large optical interferometer.

The field of light-optical Sagnac interferometry had a revived interest after the development of laser (see for instance the beautiful review paper by Post[22], where the previous experiments are carefully described and their theoretical implications analyzed). As a consequence, there was an increasing precision in measurements and a growth of technological applications, such as inertial navigation[23], where the "fiber-optical gyro"[24] and the "ring laser"[25] are used.

1.2 Universality of the Sagnac Effect

Until now, we have been speaking of the Sagnac effect for light waves. However the effect has an universality which goes beyond the nature of the interfering beams: this can be easily demonstrated and understood in SRT.

The validity of eq. (3) for any couple of counter-propagating electromagnetic beams is a very remarkable feature of the Sagnac effect, and a first important indication of its universality. In fact it shows that the effect depends only on the angular velocity of the turntable and on the path of the beams on the turntable; on the contrary, it does not depend on the light wavelength and on the presence of the (comoving) optical medium.

However, the strongest claim from its universality comes from the fact that the effect turns out to be exactly the same for any kind of "entities" (such as electromagnetic and acoustic waves, classical particles and electron Cooper pairs, neutron beams and De Broglie waves and so on...) travelling in opposite directions along a closed path in a rotating interferometer, with the same (in absolute value) velocity with respect to the turntable. Of course

the "entities" take different times for a complete round-trip, depending on their velocity relative to the turntable; *but the difference between these times is always given by eq. (3)*. For matter entities, this time difference can be obtained, for instance, using the relativistic law of velocity composition (see Malykin[26] and Rizzi-Ruggiero[27]). So, the amount of the time difference is always the same, both for matter and light waves, independently of the physical nature of the interfering beams.

This astounding but experimentally well proved fact, is the most important clue for preferring the special relativistic explanation of the Sagnac effect. In fact, its "universality" cannot be explained on the bases of the classical physics, but it can be easily explained as a "geometrical effect" in space-time, on the bases of relativistic physics. In fact, in SRT, the crucial clue leading to a geometrical (i.e. universal) explanation is the fact that the time difference between any couple of "entities" exactly coincides with (twice) the synchronization gap predicted for non-time-orthogonal physical frames (the so-called "time-lag", see f.i. Anandan[14] and Rizzi-Tartaglia[15]).

1.3 Experimental tests and derivation of the Sagnac Effect

The Sagnac effect with matter waves has been verified experimentally using Cooper pair[28] in 1965, using neutrons[29] in 1984, using ^{40}Ca atoms beams[30] in 1991 and using electrons, by Hasselbach-Nicklaus[31], in 1993. The effect of the terrestrial rotation on neutron phase was demonstrated in 1979 by Werner *et al.*[32] in a series of famous experiments.

The Sagnac phase shift has been derived, in the first order approximation, in various ways by different authors (see the paper by Hasselbach-Nicklaus quoted above, for discussion and further references), often using an heterogeneous mixture of classical kinematics and relativistic dynamics, or non relativistic quantum mechanics and some relativistic elements.

An example of derivation of the Sagnac effect for material beams, which is based on this odd mixture of non-relativistic quantum mechanics, newtonian mechanics and intrinsically relativistic elements, was given in a well known paper by Sakurai[33]. Sakurai's derivation is based on a formal analogy between the *classical* Coriolis force

$$\mathbf{F}_{Cor} = 2m_o\mathbf{v} \times \boldsymbol{\Omega} , \quad (4)$$

acting on a particle of mass m_o moving in a uniformly rotating frame, and the Lorentz force

$$\mathbf{F}_{Lor} = \frac{e}{c}\mathbf{v} \times \mathbf{B} \quad (5)$$

acting on a particle of charge e moving in a constant magnetic field \mathbf{B} .

Let us consider a beam of charged particles split into two different paths and then recombined. If S is the surface domain enclosed by the two paths, the resulting phase difference in the interference region turns out to be:

$$\Delta\Phi = \frac{e}{c\hbar} \int_S \mathbf{B} \cdot d\mathbf{S} \quad (6)$$

Therefore, $\Delta\Phi$ is different from zero when a magnetic field exists *inside* the domain enclosed by the two paths, even if the magnetic field felt by the particles along their paths is zero. This is the well known Aharonov-Bohm[34] effect¹.

By formally substituting

$$\frac{e}{c}\mathbf{B} \rightarrow 2m_o\boldsymbol{\Omega} \quad (7)$$

Sakurai shows that the phase shift (6) reduces to

$$\Delta\Phi = \frac{2m_o}{\hbar} \int \boldsymbol{\Omega} \cdot d\mathbf{S} \quad (8)$$

If $\boldsymbol{\Omega}$ is interpreted as the angular velocity vector of the uniformly rotating turntable, and \mathbf{S} as the vector associated to the area enclosed by the closed path along which two counter-propagating material beams travel, then eq. (8) can be interpreted as the Sagnac phase shift for the considered counter-propagating beams:

$$\Delta\Phi = \frac{2m_o}{\hbar} \boldsymbol{\Omega} \cdot \mathbf{S} \quad (9)$$

This result has been obtained using non relativistic quantum mechanics. The time difference corresponding to the phase difference (9), turns out to be:

$$\Delta t = \frac{\Delta\Phi}{\omega} = \frac{\hbar}{E} \Delta\Phi = \frac{\hbar}{mc^2} \Delta\Phi = \frac{2m_o}{mc^2} \boldsymbol{\Omega} \cdot \mathbf{S} \quad (10)$$

Let us point out that eq. (10) contains, un-consistently but unavoidably, some relativistic elements ($\hbar\omega = E = mc^2$). Of course in the first order

¹In the case of the Aharonov-Bohm effect, the magnetic field \mathbf{B} is zero along the trajectories of the particles, while in the Sakurai's derivation, which we are going to generalize, the angular velocity, which is the analogue of the magnetic field for particles in a rotating frames, is not null: therefore the analogy with the Aharonov-Bohm effect seems to be questionable. However, the formal analogy can be easily recovered when *the flux* of the magnetic field, rather than the magnetic field itself, is considered: this is just what we are going to do (see Section 2, below).

approximation, i.e. when the relativistic mass m coincides with the rest mass m_o eq. (10) reduces to eq. (2); that is, as we stressed before, a first order approximation for the relativistic time difference (3) associated to the Sagnac effect².

1.4 A generalization of the Sakurai's derivation

In this paper we are going to extend the simple "derivation by analogy" used by Sakurai to a fully relativistic context. To this end the Cattaneo's 1+3 splitting[35],[36],[37],[38],[39] will be adopted: it will enable us to describe the geometrodynamics of the rotating frame in a very transparent and powerful way. In particular, the Cattaneo's splitting allows to generalize the newtonian elements used by Sakurai to a relativistic context, in which also relativistic quantum mechanics can be adopted. This new approach leads to a derivation, in a self-consistent way, of the relativistic Sagnac time delay (3), whose first order approximation coincides with Sakurai's result (10). Moreover, contrary to Sakurai's claim (see footnote 7 of the paper quoted above), in our derivation it is shown that the analogy between the Sagnac phase shift and the Aharonov-Bohm phase shift holds also in relativistic quantum mechanics.

2 The phase of quantum particles in electromagnetic field and the Aharonov-Bohm Effect

Let us consider a quantum particle of (proper) mass m_o and electric charge e . If the particle is free, the associated Dirac equation is [40]

$$\left(\gamma^\mu \partial_\mu + \frac{m_o c}{\hbar}\right) \psi(x) = 0 \quad (11)$$

where $\psi(x)$ is the spinorial wave function which is the solution of (11) and $x \equiv \{x^\mu\}$ is a point in spacetime³.

In an electromagnetic field described by the 4-potential A_μ the Dirac equation is obtained by the formal substitution $\partial_\mu \rightarrow \partial_\mu - i \frac{e}{\hbar c} A_\mu$, and the wave equation becomes

$$\left[\left(\gamma^\mu \left(\partial_\mu - i \frac{e}{\hbar c} A_\mu\right) + \frac{m_o c}{\hbar}\right)\right] \psi'(x) = 0 \quad (12)$$

²Formulas (2) and (10) differs by a factor 2: this depends on the fact that in eq. (2) we considered the complete round-trip of the beams, while in this section we refer to a situation in which the emission point and the interference point are diametrically opposed.

³Let $(-1, 1, 1, 1)$ be the signature of spacetime; Greek indices run from 0 to 3, while Latin indices run from 1 to 3.

where $\psi'(x)$ is the spinorial solution of (12).

According to this formulation of the interaction between the electromagnetic field and the particle, it can be shown that, if $\psi(x)$ is a solution of a physical problem for the free quantum particle according to (11), the corresponding solution for the interacting wave equation (12) turns out to be

$$\psi'(x) = \exp\left(i\frac{e}{\hbar c} \int^x A_\mu(x') dx'^\mu\right) \psi(x) \quad (13)$$

One says that the A_μ field has produced a non-integrable phase factor that depends on the past history of the particle, which appears in (13) as the domain of integration⁴.

This analysis leads to the existence of a remarkable phenomenon.

Consider the two slits experiment (figure 1) and imagine that a single coherent charged beam is split into two parts, which travel in a region where only a magnetic field is present, described by the 3-vector potential \mathbf{A} ; then the beams are recombined to observe the interference pattern. The phase of the two wave functions, at each point of the pattern, will be modified, with respect to the case of free propagation ($\mathbf{A} = 0$), by factors of the form given in (13), which depend on the respective space trajectories. The magnetic potential-induced phase shift has the form

$$\Delta\Phi = \frac{e}{\hbar c} \left(\int_{C_1} A_i dx'^i - \int_{C_2} A_i dx'^i \right) = \frac{e}{\hbar c} \oint_C \mathbf{A} \cdot d\mathbf{r} = \frac{e}{\hbar c} \int_S \mathbf{B} \cdot d\mathbf{S} \quad (14)$$

where C is the oriented closed curve, obtained as the sum of the oriented paths C_1 and C_2 relative to each component of the beam (in the physical space, see figure 1). Eq. (14) expresses (by means of the Stoke's Theorem) the phase difference in terms of the flux of the magnetic field across the surface S enclosed by the curve C .

Aharonov and Bohm[34] applied this result to the situation in which the two split beams pass one on each side of a solenoid inserted between the paths (see figure 2). Thus, even if the magnetic field \mathbf{B} is totally contained within the solenoid, and the beams pass through a $\mathbf{B} = 0$ region, a resulting phase shift appears, since a non null magnetic flux is associated to every closed path which encloses the solenoid.

⁴This is a very general result, that applies as well to the Schrödinger wave function of an interacting non relativistic particle (see below).

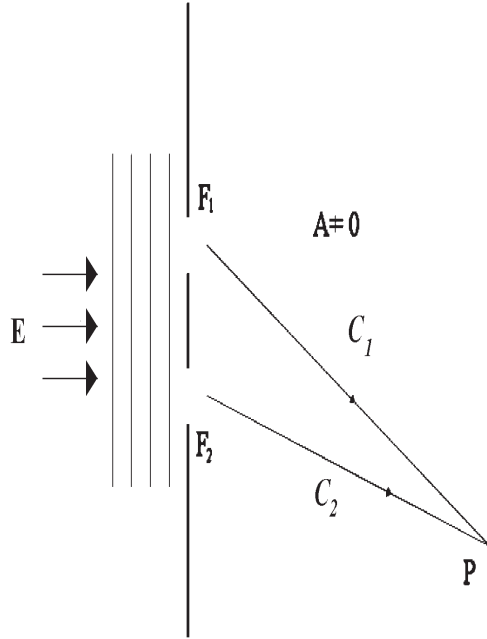


Figure 1: A single coherent charged beam, originating in E , is split into two parts (passing through the two slits F_1 and F_2) that propagate, respectively, along the paths C_1 and C_2 (in the figure these paths are represented, respectively, by EF_1P and EF_2P). The beams travel in a region where a vector potential \mathbf{A} is present. In P , the beams interfere and an additional phase shift is provoked by the magnetic field.

We need a relativistic wave equation in order to generalize the Sakurai's "derivation by analogy" to a fully relativistic context. However, Tourrenc[41] showed that no explicit wave equation is demanded to describe the Aharonov-Bohm effect, since its interpretation is a pure geometric one: in fact eq. (14) is independent of the very nature of the interfering charged beams, which can be spinorial, vectorial or tensorial. In particular, from a physical point of view, spin has no influence on the Aharonov-Bohm effect because there is no coupling with the magnetic field which is confined inside the solenoid⁵.

Things are different when a particle with spin, moving in a rotating frame, is considered. In this case a coupling between the spin and the angular velocity of the frame appears (this effect is evaluated by Hehl-Ni[42] and Mashhoon[43]). As a consequence, our formal analogy between matter waves, moving in a uniformly rotating frame and charged beams, moving in a region⁶ where a constant magnetic potential is present, holds only when the spin-rotation coupling is neglected.

3 Generalized Coriolis and Lorentz Forces

In this section we shall introduce the generalized Coriolis and Lorentz forces, which will permit us to extend to a pure relativistic context the Sakurai's procedure which we outlined in Sect. 1.3

First of all, let us choose a physical frame, which is represented in space-time by a time-like congruence Γ of world lines of the particles constituting the 3-dimensional physical frame; let $\gamma(x)$ be the field of unit vectors tangent to the world lines of the congruence Γ . Now, let us choose a system of admissible coordinate so that the lines $x^0 = var$ coincide with the lines of Γ ; according to Cattaneo's terminology, such coordinates are said to be 'adapted to the physical frame' defined by the congruence Γ .

Being $g_{\mu\nu}\gamma^\mu\gamma^\nu = -1$, the contravariant and covariant components of the γ -field are:

$$\left\{ \begin{array}{l} \gamma^o = \frac{1}{\sqrt{-g_{oo}}} \\ \gamma^i = 0 \end{array} \right. \quad \left\{ \begin{array}{l} \gamma_o = \sqrt{-g_{oo}} \\ \gamma_i = g_{io}\gamma^o \end{array} \right. \quad (15)$$

The physical spacetime is a (pseudo) riemannian manifold \mathcal{M} , and in each point $p \in \mathcal{M}$, the tangent space T_p can be split into the direct sum of two

⁵If the magnetic field is null, the Dirac equation is equivalent to the Klein-Gordon equation, and this is the case of a situation when a constant potential is present. Therefore, in what follows we shall just use eq. (14) and we shall not refer explicitly to any relativistic wave equation.

⁶In a non rotating frame.

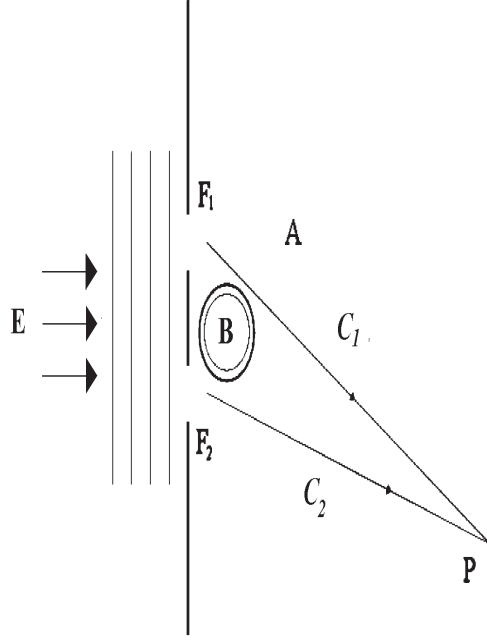


Figure 2: A single coherent charged beam, originating in E , is split into two parts (passing through the two slits F_1 and F_2) that propagate, respectively, along the paths C_1 and C_2 (in the figure these paths are represented, respectively, by EF_1P and EF_2P). Between the paths a solenoid is present; the magnetic field \mathbf{B} is entirely contained inside the solenoid, while outside there is a constant vector potential \mathbf{A} . In P , the beams interfere and an additional phase shift, provoked by the magnetic field confined inside the solenoid, is observed.

subspaces: Θ_p , spanned by γ^α , which we shall call "local time direction" of the given frame, and Σ_p , the 3-dimensional subspace which is supplementary (M-orthogonal) with respect to T_p ; Σ_p is called "local space platform" of the given frame. So, the tangent space can be written as the direct sum

$$T_p = \Theta_p \oplus \Sigma_p \quad (16)$$

A vector which belongs T_p can be projected onto Θ_p and Σ_p using, respectively, the *time projector* $\gamma_\mu \gamma_\nu$ and the *space projector* $\gamma_{\mu\nu} \doteq g_{\mu\nu} - \gamma_\mu \gamma_\nu$, which is interpreted as "spatial metric tensor". Then the "transverse" derivative operator $\tilde{\partial}_\mu \doteq \partial_\mu - \gamma_\mu \gamma^\sigma \partial_\sigma$ can be introduced (even if we shall confine ourselves only to stationary situations, in which $\partial_\sigma \equiv 0$). Finally, let us introduce the space vortex tensor of the congruence:

$$\tilde{\Omega}_{hk} \doteq \gamma_o \left[\tilde{\partial}_h \left(\frac{\gamma_k}{\gamma_o} \right) - \tilde{\partial}_k \left(\frac{\gamma_h}{\gamma_o} \right) \right] \quad (17)$$

and let $\omega(x) \in \Sigma_p$ be the axial 3-vector associated to the space vortex tensor of the congruence by means of the relation

$$\omega^i \doteq \frac{c}{4} \varepsilon^{ijk} \tilde{\Omega}_{jk} = \frac{c}{2} \varepsilon^{ijk} \gamma_o \tilde{\partial}_j \left(\frac{\gamma_k}{\gamma_o} \right) \quad (18)$$

where $\epsilon^{ijk} \doteq \frac{1}{\sqrt{\det(\gamma_{ij})}} \delta^{ijk}$ is the Ricci-Levi Civita tensor, defined in terms of the completely antisymmetric symbol δ^{ijk} and of the spatial metric tensor γ_{ij} .

The equation of motion of a particle, relative to this physical frame, can be obtained by means of the Cattaneo's projection technique. In Appendix A the general form of this equation is given, in coordinates adapted to the physical frame (see eqs. (37), (42), (43))

In particular, in eq. (43), a term which depends on the 'standard relative velocity' \mathbf{v} of the particle appears. It can be thought of as a generalized Coriolis-like force:

$$\mathcal{F}_i = 2m(\mathbf{v} \times \boldsymbol{\omega})_i \quad (19)$$

where m is the relativistic mass $m \doteq m_o \left(1 - \frac{v^2}{c^2} \right)^{-\frac{1}{2}}$ of the particle.

Now, let us introduce the "gravito-electric potential" ϕ^G and the "gravito-magnetic potential" A_i^G defined by

$$\begin{cases} \phi^G \doteq -c^2 \gamma^o \\ A_i^G \doteq c^2 \frac{\gamma_i}{\gamma_o} \end{cases} \quad (20)$$

In terms of these potentials, the vortex 3-vector ω^i is expressed in the form

$$\omega^i = \frac{1}{2c} \varepsilon^{ijk} \gamma_o \left(\tilde{\partial}_j A_k^G \right) \quad (21)$$

Alternatively, it can be written in the form

$$\omega^i = \frac{1}{2c} \gamma_o \left(\tilde{\nabla} \times \mathbf{A}_G \right)^i \doteq \frac{1}{2c} \gamma_o B_G^i \quad (22)$$

where we implicitly defined the "gravito-magnetic" field

$$B_G^i \doteq \left(\tilde{\nabla} \times \mathbf{A}_G \right)^i \quad (23)$$

In terms of this field, the velocity-dependent force (19) becomes

$$\mathcal{F}_i = m \gamma_o \left(\frac{\mathbf{v}}{c} \times \mathbf{B}_G \right)_i \quad (24)$$

which has the form of a "gravito-magnetic" Lorentz force.

Notice that the Coriolis-like force (19) transforms into the Lorentz-like force (24) with the formal substitution

$$2m\boldsymbol{\omega} \rightarrow \frac{m\gamma_o}{c} \mathbf{B}_G \quad (25)$$

4 Sagnac effect for matter waves

Now we want to apply the formal analogy described in the previous section to the phase shift induced by rotation on a beam of massive particles which, after being split, propagate in two opposite directions along the rim of a rotating disk. When they are recombined, the resulting phase shift is the manifestation of the Sagnac effect.

To this end, let us consider the analogue of the phase shift (14) for the gravito-magnetic field introduced before

$$\Delta\Phi = \frac{2m\gamma_o}{c\hbar} \oint_C \mathbf{A}^G \cdot d\mathbf{r} = \frac{2m\gamma_o}{c\hbar} \int_S \mathbf{B}^G \cdot d\mathbf{S} \quad (26)$$

which is obtained on the bases of the formal analogy between eq. (24) and the magnetic force (5):

$$\frac{e}{c} \mathbf{B} \rightarrow \frac{m\gamma_o}{c} \mathbf{B}^G \quad (27)$$

To evaluate the phase shift (26) we must consider the congruence which describes the rotating frame in spacetime. In particular, the space vectors belong to the (tangent bundle to the) "relative space" of the disk, which is the only space having an actual physical meaning from an operational point of view, and it is identified as the physical space of the rotating platform[44].

Hence, in the chart $(x^0, x^1, x^2, x^3) = (ct, r, \vartheta, z)$ adapted to the rotating frame, the covariant components of the metric tensor turn out to be[44]:

$$g_{\mu\nu} = \begin{pmatrix} -1 + \frac{\Omega^2 r^2}{c^2} & 0 & \frac{\Omega r^2}{c} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\Omega r^2}{c} & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (28)$$

where Ω is the (constant) angular velocity of rotation of the disk with respect to the laboratory frame. As a consequence, the non null components of the vector field $\gamma(x)$, evaluated on the trajectory $R = \text{const}$ along which both beams propagate, are:

$$\begin{cases} \gamma^o \doteq \frac{1}{\sqrt{-g_{oo}}} = \gamma \\ \gamma_o \doteq \sqrt{-g_{oo}} = \gamma^{-1} \\ \gamma_\vartheta \doteq g_{\vartheta o} \gamma^o = \frac{\gamma \Omega R^2}{c} \end{cases} \quad (29)$$

where $\gamma = \left(1 - \frac{\Omega^2 R^2}{c^2}\right)^{-\frac{1}{2}}$.

So, for the gravitomagnetic potential we obtain

$$A_\vartheta^G \doteq c^2 \frac{\gamma_\vartheta}{\gamma_o} = \gamma^2 \Omega R^2 c \quad (30)$$

As a consequence, the phase shift (26) becomes

$$\Delta\Phi = \frac{2m}{c\hbar\gamma} \int_0^{2\pi} A_\vartheta^G d\vartheta = \frac{2m}{c\hbar\gamma} \int_0^{2\pi} (\gamma^2 \Omega R^2 c) d\vartheta = 4\pi \frac{m}{\hbar} \Omega R^2 \gamma \quad (31)$$

According to Cattaneo's terminology, the proper time is the "standard relative time" for an observer on the rotating platform; so the proper time difference corresponding to (31) is obtained according to

$$\Delta\tau = \frac{\Delta\Phi}{\omega} = \frac{\hbar}{E} \Delta\Phi = \frac{\hbar}{mc^2} \Delta\Phi \quad (32)$$

and it turns out to be

$$\Delta\tau = 4\pi \frac{\Omega R^2 \gamma}{c^2} \equiv \frac{4\pi R^2 \Omega}{c^2 \left(1 - \frac{\Omega^2 R^2}{c^2}\right)^{1/2}} \quad (33)$$

which agrees with the proper time difference (3) due to the Sagnac effect, which, as we pointed out in subsection 1.2, corresponds to the time difference for any kind of matter entities counter-propagating in a uniformly rotating disk. As we stressed before, this time difference does not depend on the standard relative velocity of the particles and it is exactly twice the *time lag* due to the synchronization gap arising in a rotating frame.

The phase shift can be expressed also as a function of the area S of the surface enclosed by the trajectories:

$$\Delta\Phi = 2\beta^2 S\Omega \frac{m}{\hbar} \frac{\gamma^2}{\gamma - 1} = 2\frac{m}{\hbar} S\Omega (\gamma + 1) \quad (34)$$

where $\beta \doteq \frac{\Omega R}{c}$ and

$$S = \int_0^R \int_0^{2\pi} \frac{rdrd\vartheta}{\sqrt{1 - \frac{\Omega^2 r^2}{c^2}}} = 2\pi \frac{c^2}{\Omega^2} \left(1 - \sqrt{1 - \frac{\Omega^2 R^2}{c^2}} \right) = 2\pi \frac{c^2}{\Omega^2} \left(\frac{\gamma - 1}{\gamma} \right) \quad (35)$$

We notice that (34) reduces to (9)⁷, only in first order approximation with respect to $\frac{\Omega R}{c}$, i.e. when $\gamma \rightarrow 1$: the formal difference between (34) and (9) is due to the non Euclidean features of the relative space (see Rizzi-Ruggiero[44] for further details).

5 Conclusions

The Sagnac phase shift for matter waves in a uniformly rotating interferometer has been deduced, by means of a formal analogy with the magnetic potential-induced phase shift for charged particles travelling in a region where a constant vector potential is present.

The formal analogy outlined by Sakurai, which explains the effect of rotation using a "ill-assorted" mixture of non-relativistic quantum mechanics, newtonian mechanics (which are Galilei-covariant) and intrinsically relativistic elements⁸ (which are Lorentz-covariant), has been extended to a fully relativistic treatment, using the 1+3 Cattaneo's splitting technique. The space in which waves propagate has been recognized as the relative space of a rotating frame.

⁷Apart a factor 2, whose origin has been explained in the footnote 2 in Section 1.3.

⁸Indeed, the lack of self-consistency, due to the use of this odd mixture, is present not only in Sakurai's derivation, but also in all the known approaches based on the formal analogy with the Aharonov-Bohm effect.

Using this splitting technique, we have generalized the newtonian elements used by Sakurai to a fully relativistic context where we have been able to adopt relativistic quantum mechanics. In this way, we have obtained a derivation of the relativistic Sagnac time delay (whose first order approximation coincides with Sakurai's result) in a self-consistent way.

A Appendix: Equation of motion in an arbitrary physical frame

Given an arbitrary physical frame, the space projection (i.e. its "standard relative formulation") of the equation of motion

$$\frac{Dp^\alpha}{d\tau} = F^\alpha \quad (36)$$

of a particle in the external field described by the 4-vector F^α turns out to be⁹:

$$\frac{\hat{D}p_i}{dT} = m(G'_i + G''_i) + F_i \quad (37)$$

where T is the "standard relative time"; $\frac{\hat{D}}{dT}$ is a suitable derivative operator; p_i, G'_i, G''_i, F_i are "relative" space vectors and m is the relativistic mass $m \doteq m_o \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$ of the particle, in terms of its "standard relative velocity" \mathbf{v} (see Cattaneo [35],[36],[37],[38],[39]. In particular in terms of the potentials

$$\begin{cases} \phi_G \doteq -c^2 \gamma^o \\ A_{Gi} \doteq c^2 \frac{\gamma_i}{\gamma_o} \end{cases} \quad (38)$$

we can write

$$G'_i = - \left(-\tilde{\partial}_i \phi_G - \partial_o A_{Gi} \right) \quad (39)$$

which can be interpreted as a gravito-electric field:

$$E_{Gi} \doteq - \left(-\tilde{\partial}_i \phi_G - \partial_o A_{Gi} \right) \quad (40)$$

Moreover, considering that

$$G''_i = 2\varepsilon_{ijk}\omega^k v^j = \sqrt{\det(\gamma_{ij})}\delta_{ijk}\omega^k v^j \quad (41)$$

⁹The field F^α includes the possible constraints.

and the definition of the gravitomagnetic field (22), the equation of motion (37) can be written in the form

$$\frac{\hat{D}p_i}{dT} = mE_{Gi} + m\left(\frac{\mathbf{v}}{c} \times \mathbf{B}_G\right)_i + F_i \quad (42)$$

which is similar to the equation of motion of a particle acted upon by a Lorentz force and an external field. Alternatively, we can rewrite (42) using the rotation vector $\boldsymbol{\omega}$

$$\frac{\hat{D}p_i}{dT} = mE_{Gi} + 2m(\mathbf{v} \times \boldsymbol{\omega})_i + F_i \quad (43)$$

where the Coriolis-like force $\mathcal{F}_i = 2m(\mathbf{v} \times \boldsymbol{\omega})_i$ has been evidenced.

Although the most popular time+space splitting is the 1+3 ADM splitting of Arnowitt, Deser, Misner[45] (see also[46]), the simplicity of eqs. (42),(43) and their formal analogy with the "classical" equation of motion, make the Cattaneo splitting more suitable for our purposes.

For a modern formulation of the Cattaneo's splitting, and its relations with ADM splitting, see, for instance Jantzen *et al.* [47], and the references therein.

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